# Unit 11: Conic Sections and Polar Coordinates

## Precalculus Honors | Standard PC.MGSR.1

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# Unit Overview

We will analyze the behaviors of conic sections and polar coordinates to model mathematical and real-world situations.

- **Standard Form (1.1)** — Identify and graph circles, parabolas, ellipses, and hyperbolas.
- **General Form & Applications (1.2)** — Complete the square to convert equations and investigate real-world models like orbital paths and satellite dishes.
- **Polar Coordinates (1.3)** — Define polar coordinates and relate them to the Cartesian plane.

# Conic Sections in Standard Form

# The Four Conics

Each conic section is defined by its standard form equation centered at $(h, k)$:

| Conic | Standard Form Equation |
|---|---|
| **Circle** | $(x-h)^2 + (y-k)^2 = r^2$ |
| **Parabola** | $(y-k)^2 = 4p(x-h)$ or $(x-h)^2 = 4p(y-k)$ |
| **Ellipse** | $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ |
| **Hyperbola** | $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ (Horizontal) |

# Interactive: Ellipse Explorer

```
//| panel: sidebar
viewof h = Inputs.range([-5, 5], {value: 0, step: 1, label:
viewof k = Inputs.range([-5, 5], {value: 0, step: 1, label:
viewof a = Inputs.range([1, 10], {value: 5, step: 0.5, labe
viewof b = Inputs.range([1, 10], {value: 3, step: 0.5, labe
```

```
//| panel: fill
t = Array.from({length: 101}, (_, i) => (i / 100) * 2 * Mat
ellipsePoints = t.map(phi => [h + a * Math.cos(phi), k + b

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-12, 12], grid: true},
  y: {domain: [-12, 12], grid: true},
  marks: [
    Plot.line(ellipsePoints, {stroke: "#AA4499", strokeWidt
    Plot.dot([[h, k]], {fill: "#332288", r: 4}),
    Plot.text([[h, k - 1]], {text: [`Center: (${h}, ${k})`]
  ]
```

# Practice: PC.MGSR.1.1

**1. Identify the center and radius of $(x+3)^2+(y-5)^2=49$.**

Center: $(-3,5)$ Radius: $7$

**2. Is $\frac{x^2}{25}-\frac{y^2}{9}=1$ an ellipse or a hyperbola?**

Hyperbola (due to the subtraction).

**3. Find the vertex and direction of opening for $(x-2)^2=-8(y+1)$.**

Vertex: $(2,-1)$ Opens: Down (since $4p$ is negative).

**4. Find the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.**

$c^2 = a^2 - b^2 = 25 - 16 = 9 \rightarrow c = 3$ Foci: $(\pm 3, 0)$

**5. Graph the circle centered at the origin with a diameter of 10.**

$r = 5 \rightarrow x^2 + y^2 = 25$

# General Form & Real-World Modeling

# Converting to Standard Form

General Form: $Ax^2 + Cy^2 + Dx + Ey + F = 0$. To graph, we **complete the square**:

1. Group $x$ and $y$ terms; move constants to the right.
2. Factor out leading coefficients from $x^2$ and $y^2$.
3. Add $(\frac{b}{2})^2$ to both sides.
4. Factor into $(x-h)^2$ and $(y-k)^2$.

# Real-World Conics

Conic sections model critical physical behaviors:

- **Parabolas:** Satellite dishes and flashlights (reflective properties).
- **Ellipses:** Planetary orbital paths and "whispering galleries."
- **Hyperbolas:** Sonic booms and certain long-range navigation systems (LORAN).

# Practice: PC.MGSR.1.2

**1. Convert $x^2 + y^2 - 4x + 6y - 3 = 0$ to standard form.**

$(x^2 - 4x + 4) + (y^2 + 6y + 9) = 3 + 4 + 9 \rightarrow$
$(x - 2)^2 + (y + 3)^2 = 16$

**2. A satellite dish is 10ft wide and 2ft deep. Where is the focus?**

Model as $x^2 = 4py$. Point $(5, 2)$ is on the curve.
$25 = 4p(2) \rightarrow 8p = 25 \rightarrow p = 3.125$ Focus: 3.125 ft above the vertex.

**3. Identify the conic: $4x^2 - 9y^2 + 32x + 36y - 8 = 0$.**

Hyperbola (coefficients of $x^2$ and $y^2$ have opposite signs).

**4. A whispering gallery is an elliptical room. If the room is 100ft long and 60ft wide, where should two people stand to hear each other?**

Stand at the foci. $a = 50, b = 30 \rightarrow c^2 = 2500 - 900 = 1600$. $c = 40$. They should stand 40ft from the center on either side.

**5. Convert $y^2 - 8x - 4y + 12 = 0$ to standard form.**

$(y^2 - 4y + 4) = 8x - 12 + 4 \rightarrow (y - 2)^2 = 8(x - 1)$

# Polar Coordinates

# Polar vs. Cartesian

Polar coordinates represent a point as $(r, \theta)$, where $r$ is the distance from the pole and $\theta$ is the angle.

**Conversion Formulas:**

| To Cartesian $(x, y)$ | To Polar $(r, \theta)$ |
|---|---|
| $x = r\cos\theta$ | $r^2 = x^2 + y^2$ |
| $y = r\sin\theta$ | $\tan\theta = \frac{y}{x}$ |

# Interactive: Polar Plotter

```
//| panel: sidebar
viewof pol_r = Inputs.range([0, 10], {value: 5, step: 0.1,
viewof pol_deg = Inputs.range([0, 360], {value: 45, step: 5
```

```
//| panel: fill
pol_rad = pol_deg * Math.PI / 180
pol_x = pol_r * Math.cos(pol_rad)
pol_y = pol_r * Math.sin(pol_rad)

Plot.plot({
  aspectRatio: 1,
  x: {domain: [-11, 11], grid: true},
  y: {domain: [-11, 11], grid: true},
  marks: [
    // Circles for polar grid
    [2, 4, 6, 8, 10].map(d => Plot.dot([[0,0]], {r: d * 20,
    Plot.dot([[pol_x, pol_y]], {fill: "#44AA99", r: 6}),
    Plot.ruleX([0], {stroke: "#ccc"}),
    Plot.ruleY([0], {stroke: "#ccc"})
```

# Practice: PC.MGSR.1.3

**1. Convert $(4, \frac{\pi}{6})$ to Cartesian coordinates.**

$x = 4\cos(\pi/6) = 2\sqrt{3}, \quad y = 4\sin(\pi/6) = 2 \rightarrow (2\sqrt{3}, 2)$

**2. Convert $(-3, -3)$ to polar coordinates (using radians).**

$r = \sqrt{9+9} = 3\sqrt{2} \qquad \theta = \tan^{-1}(1)$ in Q3 $\rightarrow 5\pi/4$ $(3\sqrt{2}, 5\pi/4)$

**3. Convert the rectangular equation $x^2 + y^2 = 25$ to polar form.**

$r^2 = 25 \rightarrow r = 5$

**4. Convert $r = 3\csc\theta$ to rectangular form.**

$r = \frac{3}{\sin\theta} \to r\sin\theta = 3 \to y = 3$

**5. Identify the shape of the polar equation $r = 4\cos\theta$.**

$r^2 = 4r\cos\theta \to x^2 + y^2 = 4x$ $(x-2)^2 + y^2 = 4 \to$ Circle centered at $(2, 0)$ with radius 2.

# Summary of Models

| Conic/System | Real-World Context | Key Logic |
|---|---|---|
| **Parabola** | Satellite Dishes | Reflection to a focus |
| **Ellipse** | Orbits | Two-point constant distance |
| **Hyperbola** | Navigation | Constant difference distance |
| **Polar** | Radar/Navigation | Distance and angle from center |

| Conic | Standard Form Equation |
|---|---|
| **Circle** | $(x-h)^2+(y-k)^2=r^2$ |
| **Parabola** | $(y-k)^2=4p(x-h)$ or |
| | $(x-h)^2=4p(y-k)$ |
| **Ellipse** | $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ |
| **Hyperbola** | $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ (Horizontal) |